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Why It Is Computationally Harder to Reconstruct the Past Than to Predict the Future

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At first glance, it may seem that reconstructing the past is, in general, easier than predicting the future, because the past has already occurred and it has already left its traces, while the future is yet to come, and so no traces of the future are available. However, in many real-life situations, including problems from geophysics and celestial mechanics, reconstructing the past is much more computationally difficult than predicting the future. In this paper, we give an explanation of this difficulty. This explanation is given both on a formal level (as a theorem) and on the informal level (as a more intuitive explanation).

1. A PARADOXICAL FACT: IN SOME SITUATIONS, IT IS EASIER TO PREDICT THE FUTURE THAN TO RECONSTRUCT THE PAST

At first glance, it seems that the past must be easier to reconstruct than the future. At first glance, it seems like reconstructing the past must be computationally easier than predicting the future, because (a) the past is already there, with all its traces left for the researchers to pick, while (b) the future is yet to come, and it has not left any traces yet.

In reality, it is often easier to predict the future. However, in many situations, it is computationally much easier to predict the future than to reconstruct the past. For example:

- In *geophysics,* if we assume that we know the laws describing how the system changes in time, then (a) predicting the *future* is reasonably

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easy: it means that we apply these known laws to predict the values of all physical quantities in all subsequent moments of time; so, if we have enough data, we can predict what will happen in thousands and millions of years; (b) however, if we want to use these same observations to reconstruct what happened in the *past*, the results of this reconstruction become much less certain and require much more computation.

- In *celestial mechanics*, if we know the current positions, masses, and velocities of all celestial bodes, then (a) we can very accurately predict where they will be in the *future*; e.g., we can very accurately predict the future trajectory of a spaceship; (b) however, it is much more difficult to reconstruct the *past* trajectory, e.g., to reconstruct where a given meteorite has come from; even when such a reconstruction is possible (as with meteorites traced to Mars), the corresponding computations are much more complicated than the computations needed to predict the future.

How can we explain this "paradox"?

A side comment: from the commonsense viewpoint, this "paradox" is not so paradoxical after all. Above, we gave "scientific" reasons why one might expect the past to be easier to reconstruct. However, from the commonsense viewpoint, reconstructing the past is difficult. For example, the fact that the totalitarian regimes of Orwell's antiutopia (Orwell, 1963) could relatively easily suppress the past by destroying a few documents is a good indication that, in general, reconstructing the true past could be a difficult task.

Why is this a paradox? If we know the exact equations, then, in principle, predicting the future and reconstructing the past are not that different in complexity.

Let us give two examples.

Example 1. Differential equations. Physical equations are usually invariant with respect to the change in time orientation (i.e., transformation $t \to -t$). Hence, both predicting the future and reconstructing the past mean, in mathematical terms, that we integrate the same system of differential equations.

Example 2. (Simplified) linear equations. In the simplified situation, when the equations describing how the future state $f = (f_1, \ldots, f_n)$ of the system is related to its past state $p = (p_1, \ldots, p_n)$ are *linear*, $f = Ap$, or

$$f_i = \sum_{j=1}^{n} a_{ij} p_j \tag{1}$$

predicting the future means actually computing f_i from p_j, while reconstructing the past means solving the system of linear equations (1).

- For predicting the *future*, we need n multiplications and n additions to compute each of n quantities f_i that describe the future state. In total, we need $O(n^2)$ computational steps.
- There exist algorithms that solve linear systems in $O(n^\alpha)$ steps, where $\alpha < 2.5$, and it is conjectured that it may be possible to have $\alpha \approx 2$ (see, e.g., Cormen *et al.*, 1990).

Thus, the computational complexity of reconstructing the *past* is almost the same as the computational complexity of predicting the future.

Uncertainties: an informal explanation of the paradox:

- In case of exact knowledge, the tasks of predicting the future and reconstructing the past are of equal (or almost equal) computational complexity.
- Therefore, the only reason why these tasks are in reality computationally different is because the actual knowledge is *not* precise, we have *uncertainties.*

What we are planning to do. In this paper, we will show that if we take uncertainties into consideration, then reconstructing the past is indeed much more complicated than predicting the future.

We will show it on the example of the simplest possible relationship between the past and the future: the linear equation (1).

2. MOTIVATIONS FOR THE FOLLOWING DEFINITIONS

How can we describe uncertainty in p_j and f_i? Enter intervals. Measurements are never 100% precise. Thus, if as the result of measuring a certain quantity, we get a certain value $\tilde{x}$, it does not necessarily mean that the actual value x of this quantity is exactly equal to $\tilde{x}$.

For example, if a car's speedometer shows 40 m.p.h., this does not mean that the speed is exactly 40.0000 m.p.h., it simply means that the speed is equal to 40 within the accuracy of this particular measuring instrument.

The manufacturer of the measuring instrument usually supplies it with the upper bound Δ for the measurement error $\Delta x = \tilde{x} - x$; in other words, the manufacturer guarantees that $|\Delta x| \leq \Delta$.

In this case, if we have measured a quantity x and the measurement result is $\tilde{x}$, then the only information that we have about the actual value is that this actual value cannot differ from $\tilde{x}$ by more than Δ, i.e., that this actual value must be within the *interval* $[\tilde{x} - \Delta, \tilde{x} + \Delta]$.

Computations that take this interval uncertainty into consideration are called *interval computations* (see, e.g., Kearfott and Kreinovich, 1996).

Comments:

- If no such estimate is given, then for any given measurement result, we can have arbitrary actual value of x and therefore we can say nothing about the actual value. So, if we want to call something a *measurement*, some bound must be given.
- Sometimes, in addition to the upper bound for the error, we know the *probabilities* of different error values. However, in many real-life cases, we do not know these probabilities, and the upper bound Δ is the only information about the measurement error Δx that we have.
- Since we are considering the simplest case (of a linear system) anyway, in the present paper we will restrict ourselves to the simplest case when Δ is the only information.

First step toward formalization. In the problem of predicting the *future*, we measure the past values p_j and we try to reconstruct the future values f_i. Since the past values are obtained from measurements, we only know the *intervals* $\mathbf{p}_j = [\underline{p}_j, \overline{p}_j]$ of possible values of p_j.

Since we do not know the exact values of p_j, we cannot hope to predict the exact values of f_i; at best, we can hope to get some *intervals* $\mathbf{f}_i$ of possible values of f_i.

Similarly, when we reconstruct the *past*, we start with measuring the future values f_i. Thus, we start with the intervals $\mathbf{f}_i$, and we are interested in finding the intervals $\mathbf{p}_j$ of possible values of p_j.

We also need to describe uncertainties in a_{ij}. If we knew the coefficients a_{ij} precisely, then we would be able to complete the formalization. However, in many real-life situations, these values a_{ij} must also be determined from measurement, and are therefore also uncertain.

How can we describe this uncertainty? A natural way to find the values of a_{ij} is as follows:

- We prepare, very carefully, a state with the known values of parameters $p = (p_1, \ldots, p_n)$.
- Then, after a certain period of time, we measure the parameters $f_1, \ldots, f_n$ of the resulting state.

The resulting measurements have *uncertainty* in them, so, as a result, we have the *intervals* $\mathbf{f}_j$ of possible values of f_j. As a result, from equation (1), we can only get *interval* estimates for the unknown values a_{ij}.

Comment. This is where time symmetry is broken. In the idealized case when measurements are absolutely precise, the problem is symmetric with respect to time reversal: From the equation $f = Ap$ we can go to a similar equation $p = A^{-1}f$ for an inverse matrix A^{-1}.

However, under a more realistic circumstance, when we take uncertainty into consideration, the symmetry disappears. Indeed:

- We can carefully generate precise values in the *present* and trace how they evolve in the *future*.
- However, the very fact that we are generating these values right now means that before the generation, these values did not exist, and therefore their past "evolution" cannot be traced.

For example, we can very carefully place a spaceship at a given position, give it a prescribed velocity, and by measuring its trajectory, test where it goes, say, in 1 min. However, it is impossible to make an experiment in which the initial position and velocity are fixed in such a way that the position in 1 min is equal to the fixed point.

Now we are ready for the formal definitions.

3. DEFINITIONS

Definition 1. By predicting the future, we mean the following problem:

Given: n intervals $\mathbf{p}_j = [\underline{p}_j, \overline{p}_j]$, $1 \leq j \leq n$, and $n \times n$ intervals $\mathbf{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$, $1 \leq i, j \leq n$.

Find: The intervals $\mathbf{f}_i = [\underline{f}_i, \overline{f}_i]$, $1 \leq i \leq n$, of possible values of $f_i = \sum a_{ij}p_j$ when $a_{ij} \in \mathbf{a}_{ij}$ and $p_j \in \mathbf{p}_j$.

Definition 2. By reconstructing the past, we mean the following problem:

Given: n intervals $\mathbf{f}_i = [\underline{f}_i, \overline{f}_i]$, $1 \leq i \leq n$, and $n \times n$ intervals $\mathbf{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$, $1 \leq i, j \leq n$.

Find: The intervals $\mathbf{p}_j = [\underline{p}_j, \overline{p}_j]$, $1 \leq j \leq n$, of possible values of p_j, where $f_i = \sum a_{ij}p_{ij}$, $a_{ij} \in \mathbf{a}_{ij}$ and $f_i \in \mathbf{f}_i$.

4. RESULTS

Known results of interval computations show that predicting the past is indeed much more difficult. It is known that the *predicting the future* problem (described in Definition 1) requires $O(n^2)$ computational steps, while the *reconstructing the past* problem (described in Definition 2) is, in general, computationally intractable (NP-hard) (see, e.g., Rohn and Kreinovich, 1995; Kreinovich *et al.*, 1993, 1996a,b; Lakeyev and Kreinovich, 1997).

These results clearly prove that reconstructing the past is indeed a much more difficult problem than predicting the future.

Can we get an intuitive understanding of these results? The proofs of the above results are reasonably formal and not very intuitive. Since our goal is to solve a *physical* problem, we would like to have some more *intuitive* explanations for why reconstructing the past is so more difficult.

These explanations are provided in Oettli and Prager (1964) and Alefeld *et al.* (1996a, b,n.d.-a,b), which describe the geometry of the set of possible values of $p = (p_1, \ldots, p_n)$ in Definition 2. Namely, it turns out that:

- In the simplest case, the set is piecewise *linear* (Oettli and Prager, 1964).
- For symmetric matrices a_{ij}, it is piecewise *quadratic* (Alefeld *et al.*, 1996a,b, n.d.-a).
- In the general case, it can be of *arbitrary* algebraic *complexity* (Alefeld *et al.*, n.d.-b).

On the other hand, the equations that describe the set of possible values of $f = (f_1, \ldots, f_n)$ is Definition 1 are always *quadratic*.

This difference in algebraic complexity gives an intuitive explanation of why reconstructing the past is a more difficult problem than predicting the future.

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